DIFFERENTIAL CALCULUS OF EQUATIONALLY DEFINED FUNCTIONS BY WAY OF POLYNOMIAL EXPANSIONS

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0. Introduction. A calculus for "*just plain folks*", «Workshop, 1986 #35», need not be based on the notion of limits. Indeed, and it is an idea going back at least to «Lagrange, 1797 #17», functions can be studied locally from their jets, that is from their best polynomial approximations, *obtained a priori*. In «Schremmer, In Press #23», we sketched such an approach in the case of polynomial functions and in «Schremmer, In preparation #24», we will discuss rational functions and show how, in this particular case, we can even derive a certain amount of *global* information from a small number of *local* investigations.

Here, after briefly recapitulating the main features of Lagrange's approach, we discuss how it applies to functions defined by functional equations, algebraic and differential.

1. Lagrange's approach. To **expand** a function f near a given point x_0, we first **localize** it, that is we get a form in which the terms are in descending order of magnitude, by expressing $f(x_0 + h)$, the value of the function f *near* x_0, as a polynomial function $\boldsymbol{F}_{x_0}(h)$ plus a remainder $\boldsymbol{R}_{x_0}(h)$

$$f(x_0 + h) = \boldsymbol{F}_{x_0}(h) + \boldsymbol{R}_{x_0}(h),$$

where $\boldsymbol{F}_{x_0}(h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots + A_n h^n$ and $\boldsymbol{R}_n(h) = h^n$Ù$\mathbf{o}[1]$ so that $\boldsymbol{P}_{x_0}(h)$ is the **best polynomial approximation of degree *n*** of $f(x_0 + h)$. For all practical purposes, we shall just write $f(x_0 + h) = \boldsymbol{F}_{x_0}(h) + \ldots$.

We take the differential calculus to consist of "*the techniques used to find out certain properties of functions*" «Gleason, 1967 #34». The degree n of the approximation that we require depends on the nature of the required information. For instance, from a *qualitative* viewpoint, the **sign** of f *near* x_0 is given by the least non-zero approximation (usually the best constant approximation), the **variance** is given by the least non-constant approximation (usually the best affine approximation) and the **concavity** is given by the least non-affine approximation (usually the best quadratic approximation).

From a *quantitative* viewpoint, the $\boldsymbol{i}^{\mathbf{th}}$ **derivative** of f is *defined* as the function $f^{(n)}$ whose value at x_0 is $i!\,A_i$ which gives:

$$\boxed{f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)h^2/2 + f^{(3)}(x_0)h^3/3 + \ldots + f^{(n)}(x_0)h^n/n! + \ldots}$$

For instance, $[x^n]' = (n-1)x^{n-1}$ because $(n-1){x_0}^{n-1}$ is the coefficient of h in the (binomial) expansion of $(x_0 + h)^n$. Similarly, to obtain the derivative of $[\boldsymbol{f} * \boldsymbol{g}]$, where $*$ is any operation, we take the coefficient of h in the *expansion* of $[\boldsymbol{f} * \boldsymbol{g}](x_0+h)$.

That we recover the **Taylor expansion** of $\boldsymbol{f}$ should lead us to expect that C^n functions are amenable to Lagrange's approach and, in fact, the statement that "*all decent functions have continuous derivatives*" translates into "*all decent functions are practically (polynomial)*" «Gleason, 1967 #34».

The first question then is how to find the polynomial approximation. By analogy with arithmetic, it is, in the case of polynomial functions, by truncation of the high powers near 0 and of the low powers near È and, in the case of rational functions, by division in ascending powers near 0 and in descending powers near È. Anywhere inbetween requires that we first set $x = x_0 + h$.

2. Functional equations. A distinction worth making at the outset, but in fact usually not emphasized, is that, when we write $\boldsymbol{f}(x) = -3x^3 - 2x + 4$, we are defining the function $\boldsymbol{f}$ by the finite algorithm that gives the value $\boldsymbol{f}(x)$ at any point x while, when we write $\boldsymbol{g}(x) = \sqrt{x}$, we are defining the function $\boldsymbol{g}$ as solution of a functional equation, $\boldsymbol{g}^{\mathbf{2}}(x) = x$, without giving any algorithm for solving this equation and computing the value $\boldsymbol{g}(x)$. It is only in the first case that a function can truly be equated with a machine. Another fact not usually stressed is that, in most cases and even in that of rational functions, the algorithm only gives *approximate* values[1].

A practical consequence is that, when given a function such as $\boldsymbol{f}(x) = \sqrt[3]{\dfrac{x-1}{\sqrt{x^2+1}}}$, the students rarely realize that the first thing to do is to find the functional equation of which it is the solution that is, with due regard to sign considerations,

$$\boldsymbol{f}^{\mathbf{6}}(x) = \frac{(x-1)^2}{x^2+1}.$$

At this point however, we can find the value of the derivative at 0 faster and more reliably than by evaluating the derivative obtained by the usual rules. To get the Best Affine Approximation near 0, we set $\boldsymbol{f}_{\mathbf{0}}(x) = A_0 + A_1 x + \ldots$ and substitute in the functional equation:

$$\begin{aligned} [A_0 + A_1 x + \ldots]^6 &= \frac{(x-1)^2}{x^2+1} \\ {A_0}^6 + 6\,{A_1}^5 x + \ldots &= \frac{1 - 2x + \ldots}{1 + \ldots} \\ &= 1 - 2x + \ldots \end{aligned}$$

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[1] Even in arithmetic, students are rarely given the opportunity to realize, for instance, that 5^2+3 is a constructive template even if the algorithm is not actually given but that $\sqrt{5}$, or for that matter $\frac{5}{3}$ or even 5–3, is nothing but the name given a priori to the solution of an equation, $x^2 = 5$, (resp. $3x = 5$ or $x + 3 = 5$) , *should it exist*, and that, in fact, we can usually only approximate the solution. Thus, the distinction between arithmetic and algebra is somewhat counterproductive if not disingenuous.

Identifying the coefficients gives $A_0 = 1$ and $A_1 = -\sqrt[5]{\frac{1}{3}}$ so that $\boldsymbol{f_0}(x) = 1 - \sqrt[5]{\frac{1}{3}}\, x + \ldots$ and $\boldsymbol{f}'(0) = -\sqrt[5]{\frac{1}{3}}$. The equation of the tangent at the origin is, of course, $\boldsymbol{t_0}(x) = -\sqrt[5]{\frac{1}{3}}\, x + 1$.

To obtain the derivative of $\boldsymbol{f}(x)$, we need the coefficient of h at x_0. We localize:

$$\boldsymbol{f}(x_0 + h)^6 = \frac{(x_0 + h - 1)^2}{(x_0 + h)^2 + 1}$$

and expand

$$(A_0 + A_1 h + \ldots)^6 = \frac{(x_0-1)^2 + 2(x_0-1)h + \ldots}{x_0{}^2 + 1 + 2x_0 h + \ldots}$$

Dividing in ascending powers, we get

$$A_0{}^6 + 6A_1 h + \ldots = \frac{(x_0 - 1)^2}{x_0{}^2 + 1} + \frac{3x_0{}^2 - 2x_0 - 1}{x_0(x_0{}^2 + 1)}\, h + \ldots$$

Identifying the coefficients gives $A_0 = \sqrt[6]{\dfrac{(x_0 - 1)^2}{x_0{}^2 + 1}}$ and $A_1 = \dfrac{3x_0{}^2 - 2x_0 - 1}{6x_0(x_0{}^2 + 1)}$

Observe that getting the second derivative would not be that much more difficult.

12. Exponential and Logarithm Functions. There are at least three approaches to the exponential function a^x. The approach most currently favored these days is to begin by introducing the notion of integral in the middle of the differential calculus for the sole purpose of defining e^x as $\left[\int \frac{dx}{x}\right]^{-1}$, the inverse of the indefinite integral of the reciprocal function!

A more natural way would be to extend the notion of power to irrational exponents and introduce a^x as limit of a^{s_n}, where s_n is rational and approaches x as n approaches È. Unfortunately, establishing the usual computational rules is a rather forbidding exercise.

The third approache introduces a^x as solution of the initial value problem

$$f'(x) = kf(x),$$
$$f(x_0) = y_0.$$

We first consider the case $k = 1$, $x_0 = 0$.

Let $f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + ... + A_nx^n$ which we differentiate to get $f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + ... + nA_nx^{n-1}$ Substituting in the differential equation, neglecting the term A_nx^n in $f(x)$ since it is small when x is near 0 and identifying the coefficients we obtain:

$$\begin{aligned}
A_0 &= y_0 \quad \text{(from the initial condition)}\\
A_1 &= A_0\\
2A_2 &= A_1\\
3A_3 &= A_2\\
&....\\
nA_n &= A_{n-1}
\end{aligned}$$

from which we get $A_n = \dfrac{y_0}{n!}$ and $f(x) = y_0 \displaystyle\sum_{i=0}^{i=n} \frac{x^n}{n!} + ...$ that is $f(x) = f(x_0) \displaystyle\sum_{i=0}^{i=n} \frac{x^n}{n!} + ...$

It is interesting to note that many of the properties of the exponential function can be recovered from this approximation. In particular, we have an addition formula for a and b near 0 :

$$\begin{aligned}
f(a)f(b) &= y_0[1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ...]y_0[1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + ...]\\
&= y_0{}^2\Big[1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ...\\
&\qquad +b + ab + \frac{a^2b}{2!} + ...\\
&\qquad + \frac{b^2}{2!} + \frac{ab^2}{2!} + ...\\
&\qquad + \frac{b^3}{3!} + ...\Big]\\
&= y_0{}^2\Big[1 + (a + b) + \frac{(a + b)^2}{2!} + \frac{(a + b)^3}{3!} + ...\Big]\\
&= y_0 f(a+b)
\end{aligned}$$

since ' ... ' stands for finite remainders and *not* for infinite tails. We thus have

$$\begin{aligned}
f(x_0 + h) &= 1 + (x_0 + h) + \frac{(x_0 + h)^2}{2!} + \frac{(x_0 + h)^3}{3!} + ...\\
&= [1 + x_0 + \frac{x_0{}^2}{2!} + \frac{x_0{}^3}{3!} + ...][1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + ...]\\
&= f(x_0)f(h)
\end{aligned}$$

which we use to localize.

We can find an approximate solution near x_0 by localizing. Writing $\boldsymbol{f}(x_0+h) = \boldsymbol{f}_{x_0}(h)$ and since by the chain rule $\boldsymbol{f}'(x) = \boldsymbol{f}'_{x_0}(h)$, the differential problem becomes

$$\begin{aligned} &\boldsymbol{f}'_{x_0}(h) = \boldsymbol{f}_{x_0}(h) \\ &\boldsymbol{f}_{x_0}(0) = y_0 \end{aligned}$$

which gives $\boldsymbol{f}_{x_0}(h) = y_0 \sum_{i=0}^{i=n} \frac{h^n}{n!} + \ldots$

Once the existence of a solution $\boldsymbol{f}(x)$ of the initial value problem $\boldsymbol{f}'(x) = \boldsymbol{f}(x)$, $\boldsymbol{f}(0) = 1$ is assumed, the properties of $\boldsymbol{f}(x)$ are easily obtained.

1. $\boldsymbol{f}(x) \neq 0$ *for all* x. Differentiating $\boldsymbol{f}(x)\boldsymbol{f}(-x)$ we get 0 so that $\boldsymbol{f}(x)\boldsymbol{f}(-x) = c$ with $c = 1$ by the initial condition.

2. $\boldsymbol{f}(-x) = \boldsymbol{f}(x)^{-1}$.

3. *Uniqueness*. Let $\boldsymbol{g}$ be another solution. Then $[\boldsymbol{g}/\boldsymbol{f}]' = 0$ so that $\boldsymbol{g} = k\boldsymbol{f}$ for some k. From the initial condition, $k = 1$ and $\boldsymbol{g} = \boldsymbol{f}$.

4. *Positivity*. From **1.** and the differential equation, $\boldsymbol{f}'(x)$ cannot have a zero so that, by the Intermediate Value Theorem, $\boldsymbol{f}'(x)$ must keep the same sign for all x and since $\boldsymbol{f}'(0) = 1$, $\boldsymbol{f}'(x) > 0$ for all x.

5. *Increasingness*. Follows from **4.**

6. $\boldsymbol{f}(a + b) = \boldsymbol{f}(a)\boldsymbol{f}(b)$. Consider the function $\boldsymbol{g}(a + x)$. Then $\boldsymbol{g}'(a + x) = \boldsymbol{f}'(a + x) = \boldsymbol{f}(a + x) = \boldsymbol{g}(x)$, so that $\boldsymbol{g}(x) = k\boldsymbol{f}(x)$ with k such that $k = \boldsymbol{g}(0) = \boldsymbol{f}(a)$. so, $\boldsymbol{f}'(a + x) = \boldsymbol{f}(a)\boldsymbol{f}(x)$ for all x.

7. $\boldsymbol{f}(x) = e^x$. We have $\boldsymbol{f}(na) = \boldsymbol{f}(a^n)$ for all positive integer n because it is true for $n = 1$ and, assuming it for n, we have $\boldsymbol{f}((n+1)a) = \boldsymbol{f}(na + a) = \boldsymbol{f}(na)\boldsymbol{f}(a) = \boldsymbol{f}^n(a)\boldsymbol{f}(a) = \boldsymbol{f}^{n+1}(a)$. Defining $e = \boldsymbol{f}(1)$ gives $\boldsymbol{f}(n) = e^n$. Since $\boldsymbol{f}$ is strictly increasing, we have $1 < e$ from $\boldsymbol{f}(0) < \boldsymbol{f}(1)$. We also have from **2.** that $\boldsymbol{f}(-n) = \boldsymbol{f}(n)^{-1} = e^{-n}$ and the result follows.

8. *Graph*. Since $e > 1$, write $e = 1 + b$ with $b > 0$ so that $e^n = (1 + b)^n \geq 1 + nb$. Since e^x is strictly increasing, e^x å È when x å È. Finally, e^{-x} å 0 when x å È so that e^x å 0 when x å –È. This gives the qualitative look of the graph.

9. *Comparison with power functions:* lim$_{n}$åÈ $(x^n/e^x) = 0$. First show that lim$_{n}$åÈ $(n/c^n) = 0$, $c > 1$, by setting $c = 1 + b$ and observing that $(1 + b)^n \geq 1 + nb + n(n - 1)/2$Ù$b^2$ and dividing by n. Now let $\varphi(x) = x/e^n$. Then $\varphi'(x) = e^x(1 - x)/e^{2x}$ and $\varphi'(x) < 0$ when $x > 1$. Hence φ is strictly decreasing and $\varphi(x)$ å 0 when x å È. A similar proof gives lim$_{n}$åÈ $(x^n/e^x) = 0$.

The logarithm function can then be introduced as inverse of the exponential function. See [**2**] for details.

Note, however, that $\boldsymbol{Ln}(x)$ is easily approximated near $x_0 = 1$ as solution of $\boldsymbol{f}'(x) = 1/x$ with the initial condition $\boldsymbol{f}(1) = 0$ since, by the chain rule, $\boldsymbol{f}'(x)|_{x_0+h} = \boldsymbol{f}'_h(x_0 + h)$ so that if we set $\boldsymbol{f}(x_0 + h) = A_0 + A_1h + A_2h^2 + A_3h^3 + \ldots$ we have $\boldsymbol{f}'(x)|_{x_0+h} = A_1 + 2A_2h + 3A_3h^2 + \ldots$ which must then be equal to $1/[1 + h] = 1 - h + h^2 - h^3 + \ldots$, that is $A_1 + 2A_2h + 3A_3h^2 + \ldots = 1 - h + h^2 - h^3 + \ldots$ from which we get the coefficients $A_0, A_1, A_2, \ldots$. In fact, here again, we need not expand near 1 but we can expand it near any $x_0 \neq 0$ as soon as we have $\boldsymbol{f}(x_0)$. The process can therefore be iterated.

13. TRIGONOMETRIC FUNCTIONS. We can define sine and cosine as solutions of the system $f' = g$ and $g' = f$ with the initial conditions $f(0) = 0$ and $g(0) = 1$ and we can again recover the usual properties from the system. See [**2**] for the details.

In a more "physical" manner, we can also define them as solutions of $f'' = -f$ with the appropriate initial conditions: $f(0) = 1$ and $f'(0) = 0$ for the cosine and $f(0) = 0$ and $f'(0) = 1$ for the sine and we can again recover the usual properties from the system. In either case, $\pi/2$ is defined as the smallest zero of $\cos x$. See [**1**] for the details.

[**1**] R. L. Finney – D. R. Ostbey. *Elementary Differential Equations with Linear Algebra*. Addison Wesley, Reading, 1984.
[**2**] S. Lang, *Analysis I*. Addison-Wesley, Reading, 1976.
[**3**] H. Levi, *Polynomials, Power Series and Calculus*. Van Nostrand, Princeton, 1968.